On the Mazur–Ulam theorem in fuzzy n–normed strictly convex spaces

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Abstract. In this paper, we generalize the Mazur–Ulam theorem in the fuzzy real n-normed strictly convex spaces.

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1. Introduction

The theory of isometric began in the classical paper [16] by S. Mazur and S. Ulam who proved that every isometry of a real normed vector space onto another real normed vector space is a linear mapping up to translation. The property is not true for normed complex vector space(for instance consider the conjugation on $\mathbb{C}$). The hypothesis of surjectivity is essential. Without this assumption, Baker [2] proved that every isometry from a normed real space into a strictly convex normed real space is linear up to translation. A number of the mathematicians have had dealt with the Mazur–Ulam theorem.
The main theme of this paper is the proof of the Mazur–Ulam theorem in a fuzzy n-normed strictly convex space.
In 1984, Katsaras [12] defined a fuzzy norm on a linear space and at the same year Wu and Fang [24] also introduced a notion of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear space. In [4], Biswas defined and studied fuzzy inner product spaces in linear space. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [14]. In 2003, Bag and Samanta [1] modified the definition of Cheng and Mordeson [6] by removing a regular condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy norms (see [1]).
In [8, 9], Gähler introduced a new approach for a theory of 2-norm and n-norm on a linear space. In [10], Hendra Gunawan and Mashadi gave a simple way to derive an (n-1)-norm from the n-norm and realized that any n-normed space is an (n-1)-normed space. Al. Narayanan and S. Vijayabalaji have introduced the notion of fuzzy n-normed linear space in [17]. Also, S. Vijayabalaji, N. Thillaigovindan and Y. B. Jun, extended n-normed linear

spaces to fuzzy n-normed linear spaces in [23]. We mention here the papers and monographs [3, 5, 7, 11, 13, 15, 18, 19, 20, 21, 22] and [25] concerning the isometries on metric spaces.

2. Preliminaries

In this section, we state some essential definitions and results which will be needed in the sequel.

Definition 2.1. *Let X be a real linear space. A function $N : X \times \mathbb{R} \longrightarrow [0, 1]$ (the so–called fuzzy subset) is said to be a fuzzy norm on X, if for all $x, y \in X$ and all $s, t \in \mathbb{R}$:*
(N_1) $N(x, t) = 0$ *for* $t \leq 0$;
(N_2) $x = 0$ *if and only if* $N(x, t) = 1$ *for all* $t > 0$;
(N_3) $N(tx, s) = N(x, \frac{s}{|t|})$ *if* $t \neq 0$;
(N_4) $N(x + y, t + s) \geq min\{N(x, t), N(y, s)\}$;
(N_5) $N(x, .)$ *is non–decreasing function on* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(x, t) = 1$;
(N_6) *For* $x \neq 0$, $N(x, .)$ *is (upper semi) continuous on* $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(x, t)$ as the truth value of the statement "the norm of x is less than or equal to the real number t".

Definition 2.2. *Let $n \in \mathbb{N}$(natural numbers) and let X be a real vector space of dimension $d \geq n$. A real valued function $\|\bullet, ..., \bullet\|$ on $X \times ... \times X$ satisfying the following four properties:*
(1)$\|x_1, ..., x_n\| = 0$, *if and only if* $x_1, ..., x_n$ *are linearly dependent;*
(2)$\|x_1, ..., x_n\|$ *is invariant under any permutation;*
(3)$\|x_1, ..., \alpha x_n\| = |\alpha|\|x_1, ..., x_n\|$, *for any* $\alpha \in \mathbb{R}$;
(4)$\|x_1, ..., x_{n-1}, y + z\| \leq \|x_1, ..., x_{n-1}, y\| + \|x_1, ..., x_{n-1}, z\|$;
is called an n-norm on X and the pair $(X, \|\bullet, ..., \bullet\|)$, is called an n-normed space.

Definition 2.3. *Let X be a real linear space over a real field F. A fuzzy subset N of $X^n \times \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers) is called the fuzzy n-normed on X, if and only if for every $x_1, ..., x_n, x_n' \in X$:*
(nN_1) *For all* $t \in \mathbb{R}$ *with* $t \leq 0$;, $N(x_1, ..., x_n, t) = 0$;
(nN_2) *For all* $t \in \mathbb{R}$ *with* $t > 0$, $N(x_1, ..., x_n, t) = 1$, *if and only if* $x_,..., x_n$ *are linearly dependent;*
(nN_3) $N(x_1, ..., x_n, t)$ *is invariant under any permutation of* $x_1, ..., x_n$;
(nN_4) *For all* $t \in \mathbb{R}$ *with* $t > 0$, $N(x_1, ..., cx_n, t) = N(x_1, ..., x_n, \frac{t}{|c|})$, *if* $c \neq 0$, $c \in F$ *(field);*
(nN_5) *For all* $s, t \in \mathbb{R}$, $N(x_1, ..., x_n + x_n', s + t) \geq min\{N(x_1, ..., x_n, t), N(x_1, ..., x_n', s)\}$;
(N_6) $N(x_1, ..., x_n, t)$, *is left continuous and non–decreasing function of* $t \in \mathbb{R}$ *and*

$$\lim_{t\to\infty} N(x_1, ..., x_n, t) = 1;$$

In this case, the pair (X, N) is called a fuzzy n-normed linear space.

Example 2.4. *Let $(X, \|\bullet, ..., \bullet\|)$ be an n-normed space. We define*

$$N(x_1, ..., x_n, t) := \begin{cases} \frac{t}{t+\|x_1,...,x_n\|}, & \text{when } t \in \mathbb{R} \text{ with } t > 0 \text{ }, (x_1, ..., x_n) \in X \times ... \times X, \\ 0, & \text{when } t \leq 0, \end{cases}$$

Then it is easy to show that (X, N) is a fuzzy n-normed linear space.

Definition 2.5. *A fuzzy n-normed space is called strictly convex, if and only if for every $x_1, ..., x_n, x_n' \in X$ and $s, t \in \mathbb{R}$, $N(x_1, ..., x_n+x_n', s+t) = min\{N(x_1, ..., x_n, t), N(x_1, ..., x_n', s)\}$ and for any $z_1, ..., z_n \in X$, $N(x_1, ..., x_n, t) = N(z_1, ..., z_n, s)$ implies that $x_1 = z_1, ..., x_n = z_n$ and $s = t$.*

Definition 2.6. *Let (X,N) and (Y,N) be two fuzzy n-normed spaces. We call $f:(X,N)\to(Y,N)$ a fuzzy n-isometry, if and only if*

$$N(x_1-x_0,...,x_n-x_0,t)=N(f(x_1)-f(x_0),...,f(x_n)-f(x_0),t),$$

for all $x_0,x_1,...,x_n\in X$ and all $t>0$.

Definition 2.7. *Let X be a real linear space and x,y,z mutually disjoint elements of X. Then x,y and z are said to be 2-collinear if $y-z=t(x-z)$, for some real number t.*

3. Mazur–Ulam problem

In this section we prove the Mazur–Ulam theorem in the fuzzy real n-normed strictly convex spaces. From now on, let (X,N) and (Y,N) be two fuzzy n-normed strictly convex spaces and $f:(X,N)\to(Y,N)$ be a function.

Lemma 3.1. *For each $x_1,...,x_n,x_n^{'}\in X$ and $t\in\mathbb{R}$,*
(i) $N(x_1,...,x_n-x_n^{'},t)=N(x_1,...,x_n^{'}-x_n,t)$*;*
(ii) $N(x_1,...,x_i,...,x_j,...,x_n,t)=N(x_1,...,x_i+\alpha x_j,...,x_j,...,x_n,t)$*, for all $\alpha\in\mathbb{R}$;*

Proof.

$$\begin{aligned}N(x_1,...,x_n-x_n^{'},t)=N(x_1,...,(-1)(x_n^{'}-x_n),t)&=N(x_1,...,x_n^{'}-x_n,\frac{t}{|-1|})\\&=N(x_1,...,x_n^{'}-x_n,t).\end{aligned}$$

To prove (ii), assume that $s,t\in\mathbb{R}$ and $s,t>0$ and $z=\frac{1}{\alpha}x_i+x_j$. By using (i) and (nN_2) and (nN_6), we have

$$\begin{aligned}&N(x_1,...,x_i,...,x_j,...,x_n,t)\le N(x_1,...,x_i,...,x_j,...,x_n,t+s)\\&\quad=N(x_1,...,\alpha(z-x_j),...,x_j,...,x_n,t+s)\\&\quad=N(x_1,...,z-x_j,...,x_j,...,x_n,\frac{t+s}{|\alpha|})\\&\quad=min\{N(x_1,...,z,...,x_j,...,x_n,\frac{t}{|\alpha|}),N(x_1,...,x_j,...,x_j,...,x_n,\frac{s}{|\alpha|})\}\\&\quad=N(x_1,...,z,...,x_j,...,x_n,\frac{t}{|\alpha|})\\&\quad=N(x_1,...,\alpha z,...,x_j,...,x_n,t)\\&\quad=N(x_1,...,x_i+\alpha x_j,...,x_j,...,x_n,t)\\&\quad\le N(x_1,...,x_i+\alpha x_j,...,x_j,...,x_n,t+s)\\&\quad=min\{N(x_1,...,x_i,...,x_j,...,x_n,t),N(x_1,...,\alpha x_j,...,x_j,...,x_n,s)\}\\&\quad=N(x_1,...,x_i,...,x_j,...,x_n,t)\end{aligned}$$

Hence, $N(x_1,...,x_i,...,x_j,...,x_n,t)=N(x_1,...,x_i+\alpha x_j,...,x_j,...,x_n,t)$, for all $\alpha\in\mathbb{R}$. □

Lemma 3.2. *Let $x_0,x_1\in X$ be arbitrary and $t>0$. Then $u=\frac{x_0+x_1}{2}$ is the unique element of X satisfying*

$$\begin{aligned}&N(x_1-u,x_1-x_n,x_2-x_n,...,x_{n-1}-x_n,t)\\&\quad=N(x_0-x_n,x_0-u,x_2-x_n,...,x_{n-1}-x_n,t)\\&\quad=N(x_0-x_n,x_1-x_n,...,x_{n-1}-x_n,2t)\end{aligned}$$

for every $x_2,...,x_n\in X$ and u, x_0 and x_1 are 2–colinear.

Proof. Since $u = \frac{x_0+x_1}{2}$, we can write

$$\begin{aligned} x_0 - u &= x_0 - \frac{x_0+x_1}{2} = \frac{x_0}{2} - \frac{x_1}{2} = \frac{x_0+x_1-x_1}{2} - \frac{x_1}{2} \\ &= -(x_1 - \frac{x_0+x_1}{2}) = -(x_1 - u). \end{aligned}$$

Thus we conclude by the Definition 2.7 that u, x_0 and x_1 are 2–colinear.
By using Lemma 3.1, we can see that

$$\begin{aligned} &N(x_1 - u, x_1 - x_n, ..., x_{n-1} - x_n, t) \\ &\quad = N(x_1 - \frac{x_0+x_1}{2}, x_1 - x_n, ..., x_{n-1} - x_n, t) \\ &\quad = N(x_1 - x_0, x_1 - x_n, ..., x_{n-1} - x_n, 2t) \\ &\quad = N(x_0 - x_n, x_1 - x_n, ..., x_{n-1} - x_n, 2t), \end{aligned}$$

and similarly

$$\begin{aligned} &N(x_0 - x_n, x_0 - u, x_2 - x_n, ..., x_{n-1} - x_n, t) \\ &\quad = N(x_0 - x_n, x_1 - x_n, ..., x_{n-1} - x_n, 2t). \end{aligned}$$

Now, we prove the uniqueness of u.
Assume that $v \in X$, satisfies the above properties. Since v, x_0 and x_1 are 2–colinear, there exists a real number s such that $v := sx_0 + (1-s)x_1$. In view of Lemma 3.1 and Definition 2.5, we obtain

$$\begin{aligned} &N(x_0 - x_n, x_1 - x_n, ..., x_{n-1} - x_n, 2t) \\ &\quad = N(x_1 - v, x_1 - x_n, x_2 - x_n, ..., x_{n-1} - x_n, t) \\ &\quad = N(x_1 - (sx_0 + (1-s)x_1), x_1 - x_n, ..., x_{n-1} - x_n, t) \\ &\quad = N(x_1 - x_0, x_1 - x_n, ..., x_{n-1} - x_n, \frac{t}{|s|}) \\ &\quad = N(x_0 - x_n, x_1 - x_n, ..., x_{n-1} - x_n, \frac{t}{|s|}). \end{aligned}$$

So, $2t = \frac{t}{|s|}$. Since $t > 0$, $|s| = \frac{1}{2}$. Also

$$\begin{aligned} &N(x_0 - x_n, x_1 - x_n, ..., x_{n-1} - x_n, 2t) \\ &\quad = N(x_0 - x_n, x_0 - v, x_2 - x_n, ..., x_{n-1} - x_n, t) \\ &\quad = N(x_0 - x_n, x_0 - (sx_0 + (1-s)x_1), x_2 - x_n, ..., x_{n-1} - x_n, t) \\ &\quad = N(x_0 - x_n, x_0 - x_1, x_2 - x_n, ..., x_{n-1} - x_n, \frac{t}{|1-s|}) \\ &\quad = N(x_0 - x_n, x_1 - x_n, ..., x_{n-1} - x_n, \frac{t}{|1-s|}). \end{aligned}$$

So $2t = \frac{t}{|1-s|}$. Hence $\frac{1}{2} = |s| = |1-s|$ and so $s = \frac{1}{2}$. Thus we obtain that $u = v$ and this complete the proof. □

Lemma 3.3. *Let $f : (X, N) \to (Y, N)$ is a fuzzy n-isometry;*
(i) For every $x_0, x_1, x_2 \in X$, if x_0, x_1 and x_2 are 2–colinear, then $f(x_0)$, $f(x_1)$ and $f(x_2)$ are 2–colinear.
(ii) If $f(0) = 0$, then for every $z_1, ..., z_n \in X$ and $t > 0$

$$N(z_1, ..., z_n, t) = N(f(z_1), ..., f(z_n), t)$$

Proof. Since x_0, x_1 and x_2 are 2–colinear, there exists a real number s such that $x_1 - x_0 = s(x_2 - x_0)$.So, for each $x_3, ..., x_{n+1} \in X$ we have

$$\begin{aligned}
&N(f(x_1) - f(x_0), f(x_3) - f(x_0), ..., f(x_{n+1}) - f(x_0), t)\\
&\quad = N(x_1 - x_0, x_3 - x_0, ..., x_{n+1} - x_0, t)\\
&\quad = N(x_2 - x_0, x_3 - x_0, ..., x_{n+1} - x_0, \frac{t}{|s|})\\
&\quad = N(f(x_2) - f(x_0), f(x_3) - f(x_0), ..., f(x_{n+1}) - f(x_0), \frac{t}{|s|})\\
&\quad = N(s(f(x_2) - f(x_0)), f(x_3) - f(x_0), ..., f(x_{n+1}) - f(x_0), t),
\end{aligned}$$

and by definition 2.5, we conclude that $f(x_1) - f(x_0) = s(f(x_2) - f(x_0))$.
To prove the property (ii), we can write

$$\begin{aligned}
N(z_1, ..., z_n, t) &= N(z_1 - 0, ..., z_n - 0, t)\\
&= N(f(z_1) - f(0), ..., f(z_n) - f(0), t)\\
&= N(f(z_1), ..., f(z_n), t).
\end{aligned}$$

□

Theorem 3.4. *Every fuzzy n-isometry $f : (X, N) \to (Y, N)$ is affine.*

Proof. $f : (X, N) \to (Y, N)$ is affine, if the function $g : (X, N) \to (Y, N)$ defined by $g(x) = f(x) - f(0)$, is linear. Its obvious that g is an n-isometry and $g(0) = 0$. Thus, we may assume that $f(0) = 0$. Hence, it is enough to show that f is linear.
Let $x_0, x_1 \in X$. By Lemma 3.1, for every $x_2, ..., x_n \in X$ we have

$$\begin{aligned}
&N(f(x_0) - f(x_n), f(x_0) - f(\frac{x_0 + x_1}{2}), f(x_2) - f(x_n), ..., f(x_{n-1}) - f(x_n), t)\\
&\quad = N(f(x_n) - f(x_0), f(\frac{x_0 + x_1}{2}) - f(x_0), f(x_2) - f(x_0), ..., f(x_{n-1}) - f(x_0), t)\\
&\quad = N(x_n - x_0, \frac{x_0 + x_1}{2} - x_0, x_2 - x_0, ..., x_{n-1} - x_0, t)\\
&\quad = N(x_n - x_0, x_1 - x_0, x_2 - x_0, ..., x_{n-1} - x_0, 2t)\\
&\quad = N(f(x_n) - f(x_0), f(x_1) - f(x_0), f(x_2) - f(x_0), ..., f(x_{n-1}) - f(x_0), 2t)\\
&\quad = N(f(x_0) - f(x_n), f(x_1) - f(x_n), f(x_2) - f(x_n), ..., f(x_{n-1}) - f(x_n), 2t).
\end{aligned}$$

And we can obtain

$$\begin{aligned}
&N(f(x_1) - f(\frac{x_0 + x_1}{2}), f(x_1) - f(x_n), f(x_2) - f(x_n)..., f(x_{n-1}) - f(x_n), t)\\
&\quad = N(f(\frac{x_0 + x_1}{2}) - f(x_1), f(x_n) - f(x_1), f(x_2) - f(x_1), ..., f(x_{n-1}) - f(x_1), t)\\
&\quad = N(\frac{x_0 + x_1}{2} - x_1, x_n - x_1, x_2 - x_1, ..., x_{n-1} - x_1, t)\\
&\quad = N(x_0 - x_1, x_n - x_1, x_2 - x_1, ..., x_{n-1} - x_1, 2t)\\
&\quad = N(f(x_0) - f(x_1), f(x_n) - f(x_1), f(x_2) - f(x_1), ..., f(x_{n-1}) - f(x_1), 2t)\\
&\quad = N(f(x_0) - f(x_n), f(x_1) - f(x_n), f(x_2) - f(x_n), ..., f(x_{n-1}) - f(x_n), 2t).
\end{aligned}$$

By (i) of Lemma (3.3), we obtain that $f(\frac{x_0+x_1}{2})$, $f(x_0)$ and $f(x_1)$ are 2–colinear. Now, from Lemma 3.2, we have

$$f(\frac{x_0 + x_1}{2}) = \frac{f(x_0)}{2} + \frac{f(x_1)}{2}$$

for all $x_0, x_1 \in X$. It follows that f is $\mathbb{Q}$-linear($\mathbb{Q}$ is the set of rational numbers). We have to show that f is $\mathbb{R}$-linear.
Let $r \in \mathbb{R}^+$ and $x \in X$. By (i) of Lemma (3.3), $f(0)$, $f(x)$ and $f(rx)$ are 2–colinear. Since

$f(0)=0$,there exists $s\in\mathbb{R}$ such that $f(rx)=sf(x)$. From (ii) of Lemma (3.3), for every $x_1,...,x_{n-1}$ and $t>0$, we have

$$\begin{aligned}
N(x,x_1,x_2,...,x_{n-1},\frac{t}{r}) &= N(rx,x_1,...,x_{n-1},t)\\
&= N(f(rx),f(x_1),f(x_2),...,f(x_{n-1}),t)\\
&= N(sf(x),f(x_1),f(x_2),...,f(x_{n-1}),t)\\
&= N(f(x),f(x_1),f(x_2),...,f(x_{n-1}),\frac{t}{|s|})\\
&= N(x,x_1,x_2,...,x_{n-1},\frac{t}{|s|}).
\end{aligned}$$

Hence $s=\pm r$. The proof is completed if $s=r$. If $s=-r$, that is, $f(rx)=-rf(x)$. Then there exists $q_1,q_2\in\mathbb{Q}$ such that $0<q_1<r<q_2$. For each $z_1,...,z_n\in X$, we have

$$\begin{aligned}
&N(f(x),f(z_1)-f(q_2x),...,f(z_{n-1})-f(q_2x),\frac{t}{q_2+r})\\
&\quad= N(q_2f(x)-(-rf(x)),f(z_1)-f(q_2x),...,f(z_{n-1})-f(q_2x),t)\\
&\quad= N(f(rx)-f(q_2x),f(z_1)-f(q_2x),...,f(z_{n-1})-f(q_2x),t)\\
&\quad= N(rx-q_2x,z_1-q_2x,...,z_{n-1}-q_2x,t)\\
&\quad= N(x,z_1-q_2x,...,z_{n-1}-q_2x,\frac{t}{q_2-r})\\
&\quad\geq N(x,z_1-q_2x,...,z_{n-1}-q_2x,\frac{t}{q_2-q_1})\\
&\quad= N(q_1x-q_2x,z_1-q_2x,...,z_{n-1}-q_2x,t)\\
&\quad= N(f(q_1x)-f(q_2x),f(z_1)-f(q_2x),...,f(z_{n-1})-f(q_2x),t)\\
&\quad= N(f(x),f(z_1)-f(q_2x),...,f(z_{n-1})-f(q_2x),\frac{t}{q_2-q_1}).
\end{aligned}$$

By (nN_6), we have $q_2+r\leq q_2-q_1$ which is a contradiction. Hence $s=r$, that is, $f(rx)=rf(x)$ for all positive real numbers r. Therefore f is $\mathbb{R}$-linear, as desired. □

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